Explicit Formulae for Bounds on the Ranks of Tilt Destabilizers and Practical Methods for Finding Pseudowalls

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1 Introduction

[ref] shows that for any rational β_0, the vertical line $\{\sigma_{\alpha,\beta_0} \colon \alpha \in \mathbb{R}_{>0}\}$ only intersects finitely many walls. A consequence of this is that if β_- is rational, then there can only be finitely many circular walls to the left of the vertical wall $\beta = \mu$. On the other hand, when β_- is not rational, [ref] showed that there are infinitely many walls.

This dichotomy does not only hold for real walls, realised by actual objects in $\mathcal{D}^b(X)$, but also for pseudowalls. Here pseudowalls are defined as 'potential' walls, induced by hypothetical Chern characters of destabilizers which satisfy certain numerical conditions which would be satisfied by any real destabilizer, regardless of whether they are realised by actual semistabilizers in $\mathcal{D}^b(X)$.

Since real walls are a subset of pseudowalls, the irrational β_- case follows immediately from the corresponding case for real walls. However, the rational β_- case involves showing that the following conditions only admit finitely many solutions (despite the fact that the same conditions admit infinitely many solutions when β_- is irrational).

For a destabilizing sequence $E \hookrightarrow F \twoheadrightarrow G$ in $\mathcal{B}^\beta$ we have the following conditions. There are some Bogomolov-Gieseker type inequalities: $0 \leq \Delta(E), \Delta(G)$ and $\Delta(E) + \Delta(G) \leq \Delta(F)$. We also have a condition relating to the tilt category $\mathcal{B}^\beta$: $0 \leq \mathrm{ch}_1^\beta(E) \leq \mathrm{ch}_1^\beta(F)$. Finally, there's a condition ensuring that the radius of the circular wall is strictly positive: $\mathrm{ch}_2^{\beta_-}(E) > 0$.

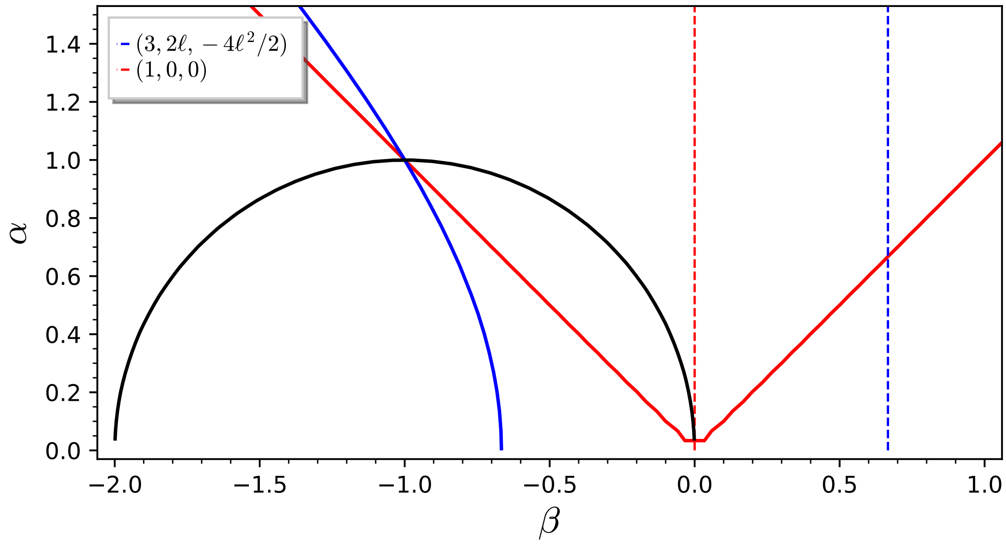


Figure 1

For any fixed $\mathrm{ch}_0(E)$, the inequality $0 \leq \mathrm{ch}_1^\beta(E) \leq \mathrm{ch}_1^\beta(F)$, allows us to bound $\mathrm{ch}_1(E)$. Then, the other inequalities allow us to bound $\mathrm{ch}_2(E)$. The final part to showing the finiteness of pseudowalls would be bounding $\mathrm{ch}_0(E)$. This has been hinted at in [ref] and done explicitly by Benjamin Schmidt within a computer program which computes pseudowalls. Here we discuss these bounds in more detail, along with the methods used, followed by refinements on them which give explicit formulae for tighter bounds on $\mathrm{ch}_0(E)$ of potential destabilizers E of F.

2 Characteristic Curves of Stability Conditions Associated to Chern Characters

TODO:

Talk about figure 1.

use Fig1

to show that given other conditions, $ch_2^{\beta-}(u) > 0 \iff$ radius > 0

3 Loose Bounds on $\mathrm{ch}_0(E)$ for Semistabilizers Along Fixed $\beta \in \mathbb{Q}$

Definition 3.1 (Twisted Chern Character)**.** *For a given β, define the twisted Chern character as follows.*

$$\mathrm{ch}^\beta(E) = \mathrm{ch}(E) \cdot \exp(-\beta \ell)$$

Component-wise, this is:

$$\begin{aligned}
\mathrm{ch}_0^\beta(E) &= \mathrm{ch}_0(E) \\
\mathrm{ch}_1^\beta(E) &= \mathrm{ch}_1(E) - \beta \,\mathrm{ch}_0(E) \\
\mathrm{ch}_2^\beta(E) &= \mathrm{ch}_2(E) - \beta \,\mathrm{ch}_1(E) + \frac{\beta^2}{2} \mathrm{ch}_0(E)
\end{aligned}$$

$\mathrm{ch}_1^\beta(E)$ is the imaginary component of the central charge $\mathcal{Z}_{\alpha,\beta}(E)$ and any element of $\mathcal{B}^\beta$ satisfies $\mathrm{ch}_1^\beta \geq 0$. This, along with additivity gives us, for any destabilizing sequence [ref]:

$$0 \leq \mathrm{ch}_1^\beta(E) \leq \mathrm{ch}_1^\beta(F) \tag{1}$$

When finding Chern characters of potential destabilizers E for some fixed Chern character $\mathrm{ch}(F)$, this bounds $\mathrm{ch}_1(E)$.

The Bogomolov form applied to the twisted Chern character is the same as the normal one. So $0 \leq \Delta(E)$ yields:

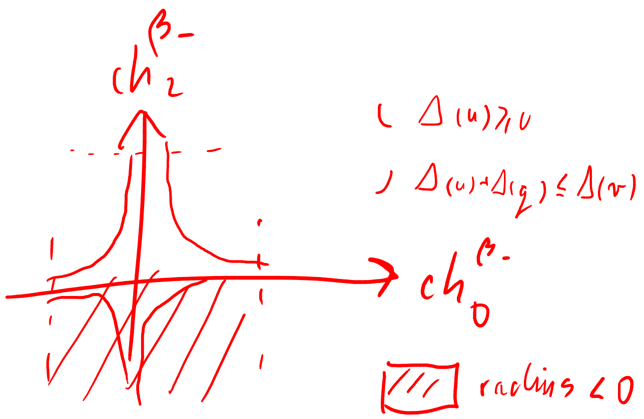
$$2\,\mathrm{ch}_0^\beta(E)\,\mathrm{ch}_2^\beta(E) \leq \mathrm{ch}_1^\beta(E)^2 \tag{2}$$

Theorem 3.1 (Bound on r - Benjamin Schmidt)**.** *Given a Chern character v such that $\beta_-(v) \in \mathbb{Q}$, the rank r of any semistabilizer E of some $F \in \mathcal{B}^\beta$ with $\mathrm{ch}(F) = v$ is bounded above by:*

$$r \leq \frac{mn^2\,\mathrm{ch}_1^\beta(v)^2}{\gcd(m, 2n^2)}$$

Proof. The restrictions on $\mathrm{ch}_0^\beta(E)$ and $\mathrm{ch}_2^\beta(E)$ is best seen with the following graph:

This is where the rationality of β_- comes in. If $\beta_- = \frac{*}{n}$ for some $*, n \in \mathbb{Z}$. Then $\mathrm{ch}_2^\beta(E) \in \frac{1}{\mathrm{lcm}(m,2n^2)}\mathbb{Z}$ where m is the integer which guarantees $\mathrm{ch}_2(E) \in$



$\frac{1}{m}\mathbb{Z}$ (determined by the variety). In particular, since $\operatorname{ch}_2(E) > 0$ we must also have $\operatorname{ch}_2^\beta(E) \geq \frac{1}{\operatorname{lcm}(m,2n^2)}$, which then in turn gives a bound for the rank of E:

$$\begin{aligned}
\operatorname{ch}_0(E) &= \operatorname{ch}_0^\beta(E) && (3)\\
&\leq \frac{\operatorname{lcm}(m,2n^2)\operatorname{ch}_1^\beta(E)^2}{2} && (4)\\
&\leq \frac{mn^2\operatorname{ch}_1^\beta(F)^2}{\gcd(m,2n^2)} && (5)
\end{aligned}$$

□

4 B.Schmidt's Method

5 Limitations

compare performance

	B. Schmidt	Me
$3\ell, -4\frac{\ell^2}{2}$	20s	instant
$3\ell, -15\frac{\ell^2}{2}$	> 2hrs	instant

reasons:
- all (r,c) with $r \leq r_{max}$ checked despite most have no sols

6 Refinement

clarified between β. and generic β

To get tighter bounds on the rank of destabilizers E of some F with some fixed Chern character, we will need to consider each of the values which $\operatorname{ch}_1^\beta(E)$ can take. Doing this will allow us to eliminate possible values of $\operatorname{ch}_0(E)$ for which each $\operatorname{ch}_1^\beta(E)$ leads to the failure of at least one of the inequalities. As opposed to only eliminating possible values of $\operatorname{ch}_0(E)$ for which all corresponding $\operatorname{ch}_1^\beta(E)$ fail one of the inequalities (which is what was implicitly happening before).

First, let us fix a Chern character for F, $\operatorname{ch}(F) = (R, C, D)$, and consider the possible Chern characters $\operatorname{ch}(E) = (r, c, d)$ of some semistabilizer E.

Recall [ref] that $\operatorname{ch}_1^\beta$ has fixed bounds in terms of $\operatorname{ch}(F)$, and so we can write:

$$c = \operatorname{ch}_1(E) = \beta r + q \qquad 0 \leq q \leq \operatorname{ch}_1^\beta(F) \tag{6}$$

Furthermore, $\operatorname{ch}_1 \in \mathbb{Z}$ so we only need to consider $q \in \frac{1}{n}\mathbb{Z} \cap [0, \operatorname{ch}_1^\beta(F)]$. For the next subsections, we consider q to be fixed with one of these values, and we shall be varying $\operatorname{ch}_0(E) = r$ to see when certain inequalities fail.

6.1 Numerical inequalities

~~6.1~~ $\Delta(E) + \Delta(G) \leq \Delta(F)$

This condition expressed in terms of R, C, D, r, c, d looks as follows:

$$0 \leq 2\,Cc - 2\,c^2 - 2\,Rd - 2\,Dr + 4\,dr \tag{7}$$

Expressing c in terms of q as defined in (eqn 6) we get the following:

$$0 \leq -2\,(\beta r + q)^2 + 2\,(\beta r + q)C - 2\,Rd - 2\,Dr + 4\,dr \tag{8}$$

This can be rearranged to express a bound on d as follows:

$$\begin{aligned} d \geq \frac{1}{4}\,R\beta^2 - \frac{R^2\beta^2}{4\,(R-2\,r)} + \frac{1}{2}\,\beta^2 r - \frac{1}{2}\,C\beta + \frac{CR\beta}{2\,(R-2\,r)} \\ - \frac{R\beta q}{R-2\,r} + \beta q + \frac{1}{2}\,D - \frac{DR}{2\,(R-2\,r)} + \frac{Cq}{R-2\,r} - \frac{q^2}{R-2\,r} \end{aligned} \tag{9}$$

Viewing equation 9 as a lower bound for d given as a function of r, the terms can be rewritten as follows. The constant term in r is $\mathrm{ch}_2^\beta(F)/2 + \beta q$. The linear term in r is $\frac{1}{2}\,\beta^2 r$. Finally, there's an hyperbolic term in r which tends to 0 as $r \to \infty$, and can be written: $\frac{R\,\mathrm{ch}_2^\beta(F)/2 + R\beta q - Cq + q^2}{2r-R}$. In the case $\beta = \beta_-$ (or β_+) we have $\mathrm{ch}_2^\beta(F) = 0$, so some of these expressions simplify.

~~6.2~~ $\Delta(E) \geq 0$

This condition expressed in terms of R, C, D, r, c, d looks as follows:

$$0 \leq c^2 - 2\,dr \tag{10}$$

Expressing c in terms of q as defined in (eqn 6) we get the following:

$$0 \leq (\beta r + q)^2 - 2\,dr \tag{11}$$

This can be rearranged to express a bound on d as follows:

$$d \leq \frac{1}{2}\,\beta^2 r + \beta q + \frac{q^2}{2\,r} \tag{12}$$

Viewing equation 12 as a lower bound for d in term of r again, there's a constant term βq, a linear term $\frac{1}{2}\,\beta^2 r$, and a hyperbolic term $\frac{q^2}{2\,r}$. Notice that for $\beta = \beta_-$ (or β_+), that is when $\mathrm{ch}_2^\beta(F) = 0$, the constant and linear terms match up with the ones for the bound found for d in subsection 6.1.

~~6.3~~ $\Delta(G) \geq 0$

This condition expressed in terms of R, C, D, r, c, d looks as follows:

$$0 \leq C^2 - 2\,DR - 2\,Cc + c^2 + 2\,Rd + 2\,Dr - 2\,dr \tag{13}$$

Expressing c in terms of q as defined in (eqn 6) we get the following:

$$0 \leq (\beta r + q)^2 - 2\,(\beta r + q)C + C^2 - 2\,DR + 2\,Rd + 2\,Dr - 2\,dr \tag{14}$$

This can be rearranged to express a bound on d as follows:

$$\begin{aligned} d \leq\; & \frac{1}{2}\,R\beta^2 - \frac{R^2\beta^2}{2\,(R-r)} + \frac{1}{2}\,\beta^2 r - C\beta + \frac{CR\beta}{R-r} - \frac{R\beta q}{R-r} \\ & + \beta q + D - \frac{C^2}{2\,(R-r)} + \frac{Cq}{R-r} - \frac{q^2}{2\,(R-r)} \end{aligned} \tag{15}$$

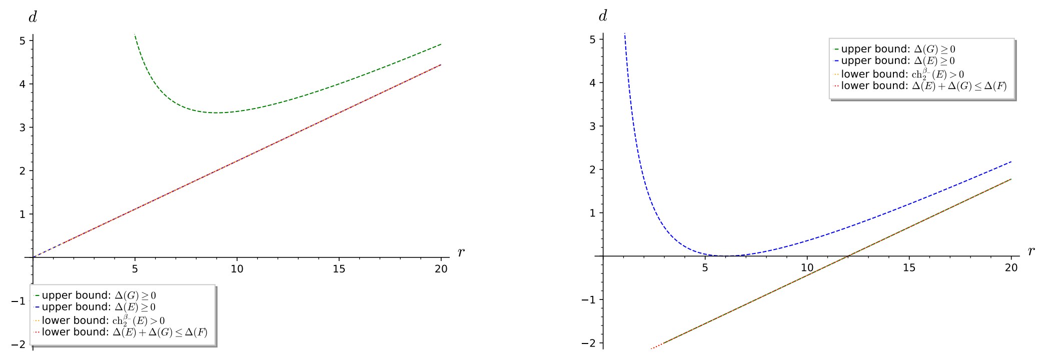
Viewing equation 15 as an upper bound for d give: as a function of r, the terms can be rewritten as follows. The constant term in r is $\mathrm{ch}_2^\beta(F) + \beta q$. The linear term in r is $\frac{1}{2}\,\beta^2 r$. Finally, there's an hyperbolic term in r which tends to 0 as $r \to \infty$, and can be written: $\frac{R\,\mathrm{ch}_2^\beta(F) + (C-q)^2/2 + R\beta q - DR}{r-R}$. In the case $\beta = \beta_-$ (or β_+) we have $\mathrm{ch}_2^\beta(F) = 0$, so some of these expressions simplify, and in particular, the constant and linear terms match those of the other bounds in the previous subsections.

~~6.4~~ Bounds on r

Now, the inequalities from the last three subsections will be used to find, for each given $q = \mathrm{ch}_1^\beta(E)$, how large r needs to be in order to leave no possible solutions for d. At that point, there are no Chern characters (r, c, d) that satisfy all inequalities to give a pseudowall.

6.2 ~~6.4.1~~ All circular pseudowalls left of vertical wall $(\beta = \beta_-)$

Suppose we take $\beta = \beta_-$ (so that $\mathrm{ch}_2^\beta(F) = 0$) in the previous subsections, to find all circular walls to the left of the vertical wall (TODO as discussed in ref).



(a) $q = 0$ (all bounds other than green coincide on line)

(b) $q = \mathrm{ch}^{\beta_-}(F)$ (all bounds other than blue coincide on line)

Figure 2: Bounds on $d := \mathrm{ch}_2(E)$ in terms of $r := \mathrm{ch}_0(E)$ for fixed, extreme, values of $q := \mathrm{ch}_1^{\beta_-}(E)$. Where $\mathrm{ch}(F) = (3, 2, -2)$.

$$d \geq \frac{1}{2}{\beta_-}^2 r + \beta_- q + -\frac{R\beta_- q - Cq + q^2}{R - 2r}, \quad \text{when } r > \frac{R}{2} \tag{16}$$

$$d \leq \frac{1}{2}{\beta_-}^2 r + \beta_- q + \frac{q^2}{2r}, \quad \text{when } r > 0 \tag{17}$$

$$d \leq \frac{1}{2}{\beta_-}^2 r + \beta_- q + -\frac{2R\beta_- q + (C - q)^2 - 2DR}{2(R - r)}, \quad \text{when } r > R \tag{18}$$

Furthermore, we get an extra bound for d resulting from the condition that the radius of the circular wall must be positive. As discussed in (TODO ref), this is equivalent to $\mathrm{ch}_2^{\beta_-}(E) > 0$, which yields:

$$d > \frac{1}{2}{\beta_-}^2 r + \beta_- q \tag{19}$$

Recalling that $q := \mathrm{ch}_1^{\beta_-}(E) \in [0, \mathrm{ch}_1^{\beta_-}(F)]$, it's worth noting that the extreme values of q in this range lead to the tightest bounds on d, as illustrated in figure (2). In fact, in each case, one of the weak upper bounds coincides with one of the weak lower bounds, (implying no possible destabilizers E with $\mathrm{ch}_0(E) =: r > R := \mathrm{ch}_0(F)$ for these q-values). This indeed happens in general since the right hand sides of (eqn 17) and (eqn 19) match when

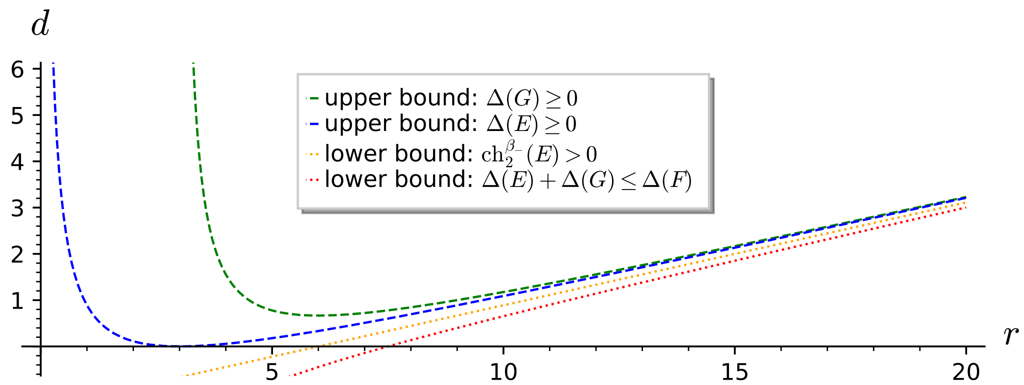


Figure 3: Bounds on $d := \mathrm{ch}_2(E)$ in terms of $r := \mathrm{ch}_0(E)$ for a fixed value $\mathrm{ch}_1^{\beta_-}(F)/2$ of $q := \mathrm{ch}_1^{\beta_-}(E)$. Where $\mathrm{ch}(F) = (3, 2, -2)$.

$q = 0$. In the other case, $q = \mathrm{ch}_1^{\beta_-}(F)$, it's the right hand sides of (eqn 18) and (eqn 19) which match.

The more generic case, when $0 < q := \mathrm{ch}_1^{\beta_-}(E) < \mathrm{ch}_1^{\beta_-}(F)$ for the bounds on d in terms of r is illustrated in figure (3). The question of whether there are pseudo-destabilizers of arbitrarily large rank, in the context of the graph, comes down to whether there are points $(r, d) \in \mathbb{Z} \oplus \frac{1}{m}\mathbb{Z}$ (with large r) that fit above the yellow line (ensuring positive radius of wall) but below the blue and green (ensuring $\Delta(E), \Delta(G) > 0$). These lines have the same assymptote at $r \to \infty$ (eqns 17, 18, 19). As mentioned in the introduction (sec 1), the finiteness of these solutions is entirely determined by whether β_- is rational or irrational. Some of the details around the associated numerics are explored next.

6.2.* **Rational** $\beta_- \neq 0$

The strategy here is similar to what was shown in (sect 3.1),

Suppose $\beta_- = \frac{a_F}{n}$ for some coprime $n \in \mathbb{N}, a_F \in \mathbb{Z}$. Then fix a value of q:

$$q := \mathrm{ch}_1^{\beta_-}(E) = \frac{b_q}{n} \in \frac{1}{n}\mathbb{Z} \cap [0, \mathrm{ch}_1^{\beta_-}(F)] \tag{20}$$

as noted at the beginning of this section (6).

Substituting the current values of q and β_- into the condition for the radius of the pseudo-wall being positive (eqn 19) we get:

$$\frac{1}{m}\mathbb{Z} \ni \qquad d > \frac{(a_F r + 2\, b_q) a_F}{2\, n^2} \qquad \in \frac{1}{2n^2}\mathbb{Z} \tag{21}$$

Theorem 6.1 (Bound on r #1)**.** *Let $v = (R, C, D)$ be a fixed Chern character. Then the ranks of the pseudo-semistabilizers for v are bounded above by the following expression.*

$$\frac{\operatorname{lcm}(m, 2n^2)}{2} \max_{q \in [0, \operatorname{ch}_1^{\beta_-}(v)]}$$
$$\left\{ \min \left(q^2, 2R\beta_- q + C^2 - 2DR - 2Cq + q^2 + \frac{R}{\operatorname{lcm}(m, 2n^2)} \right) \right\}$$

Proof. Both d and the lower bound in (eqn 21) are elements of $\frac{1}{\operatorname{lcm}(m,2n^2)}\mathbb{Z}$. So, if any of the two upper bounds on d come to within $\frac{1}{\operatorname{lcm}(m,2n^2)}$ of this lower bound, then there are no solutions for d.

Considering equations 17, 18, 19, this happens when:

$$\min \left(\frac{q^2}{2\, r}, -\frac{2\, R\beta_- q + (C - q)^2 - 2\, DR}{2\, (R - r)} \right) < \epsilon_F := \frac{1}{\operatorname{lcm}(m, 2n^2)} \tag{22}$$

This is equivalent to:

need to factor by $(q-C)$

$$r > \min \left(\frac{q^2}{2\, \epsilon_F}, \frac{2\, R\beta_- q + C^2 - 2\, DR + 2\, R\epsilon_F - 2\, Cq + q^2}{2\, \epsilon_F} \right) \tag{23}$$

If this condition holds for all q, then there are no solutions for d, and therefore r cannot satisfy this condition for all q. Taking the maximum of all these expressions over q, and substituting the value for ϵ_F gives the result. □

This bound can be refined a bit more by considering restrictions from the possible values that r take. Furthermore, the proof of theorem 6.1 uses the fact that, given an element of $\frac{1}{2n^2}\mathbb{Z}$, the closest non-equal element of $\frac{1}{m}\mathbb{Z}$ is at least $\frac{1}{\operatorname{lcm}(m,2n^2)}$ away. However this a conservative estimate, and a larger gap can sometimes be guaranteed if we know this value of $\frac{1}{2n^2}\mathbb{Z}$ explicitly.

The expressions that will follow will be a bit more complicated and have more parts which depend on the values of q and β_-, even their numerators

a_F, b_q specifically. The upcoming theorem (TODO ref) is less useful as a 'clean' formula for a bound on the ranks of the pseudo-semistabilizers, but has a purpose in the context of writing a computer program to find pseudo-semistabilizers. Such a program would iterate through possible values of q, then iterate through values of r within the bounds (dependent on q), which would then determine c, and then find the corresponding possible values for d.

Firstly, we only need to consider r-values for which $c := \mathrm{ch}_1(E)$ is integral:

$$c = \frac{a_F r}{n} + \frac{b_q}{n} \in \mathbb{Z} \tag{24}$$

That is, $r \equiv -{a_F}^{-1} b_q \mod n$ (a_F is coprime to n, and so invertible mod n). Let ${a_F}'$ be an integer representative of ${a_F}^{-1}$ in $\mathbb{Z}/n\mathbb{Z}$.

Next, we seek to find a larger ϵ to use in place of ϵ_F in the proof of theorem 6.1:

Lemma/Definition 6.1 (Finding better alternatives to ϵ_F: $\epsilon_{q,1}$ and $\epsilon_{q,2}$)**.** *Suppose $d \in \frac{1}{m}\mathbb{Z}$ satisfies the condition in eqn 21. That is:*

$$d > \frac{(a_F r + 2\, b_q) a_F}{2\, n^2}$$

Then we have:

$$d - \frac{(a_F r + 2b_q) a_F}{2n^2} \geq \epsilon_{q,2} \geq \epsilon_{q,1} > 0$$

Where $\epsilon_{q,1}$ and $\epsilon_{q,2}$ are defined as follows:

$$\epsilon_{q,1} := \frac{k_q^1}{2mn^2} \qquad \epsilon_{q,2} := \frac{k_q^2}{2mn^2}$$

where k_q^1 is the least $k \in \mathbb{Z}_{>0}$ s.t. : $k \equiv -a_F b_q m \mod n$

k_q^2 *is the least $k \in \mathbb{Z}_{>0}$ s.t. :* $k \equiv a_F b_q m (a_F {a_F}' - 2) \mod n \gcd(2n, {a_F}^2 m)$

It is worth noting that $\epsilon_{q,2}$ is potentially larger than $\epsilon_{q,2}$ but calculating it involves a gcd, a modulo reduction, and a modulo n inverse, for each q considered.

Proof. Consider the following [illegible]:

$$
\begin{aligned}
& \frac{x}{m} - \frac{(a_F r + 2b_q)a_F}{2n^2} = \frac{k}{2mn^2} && \text{for some } x \in \mathbb{Z} && (25)\\
\iff \quad & -(a_F r + 2b_q)a_F m \equiv k && \bmod 2n^2 && (26)\\
\iff \quad & -{a_F}^2 mr - 2a_F b_q m \equiv k && \bmod 2n^2 && (27)\\
\implies \quad & {a_F}^2 {a_F}' b_q m - 2a_F b_q m \equiv k && \bmod \gcd(2n^2, {a_F}^2 mn) && (28)\\
\implies \quad & -a_F b_q m \equiv k && \bmod n && (29)
\end{aligned}
$$

In our situation, we want to find the least k satisfying eqn 25. Since such a k must also satisfy eqn 29, we can pick the smallest $k_q^1 \in \mathbb{Z}_{>0}$ which satisfies this new condition (a computation only depending on q and β_-, but not r). We are then guaranteed that the gap $\frac{k}{2mn^2}$ is at least $\epsilon_{q,1}$. Furthermore, k also satisfies eqn 28 so we can also pick the smallest $k_q^2 \in \mathbb{Z}_{>0}$ satisfying this condition, which also guarantees that the gap $\frac{k}{2mn^2}$ is at least $\epsilon_{q,2}$. □

Theorem 6.2 (Bound on r #3)**.** *Let $v = (R, C, D)$ be a fixed Chern character. Then the ranks of the pseudo-semistabilizers for v with $\mathrm{ch}_1^{\beta_-} = q = \frac{a_q}{n}$ are bounded above by the following expression (with $i = 1$ or 2).*

$$\frac{1}{2\epsilon_{q,i}} \min\left(q^2, 2R\beta_- q + C^2 - 2DR - 2Cq + q^2 + \frac{R}{\operatorname{lcm}(m, 2n^2)} + R\epsilon_{q,i} \right)$$

Where $\epsilon_{q,i}$ is defined as in definition/lemma 6.1.

6.2.* **Irrational β_-**

• inf solutions

• generate with Pell-like method

• link to Yosh-Yana ?

~~7 Conclusion~~

6.3 $\beta \neq \beta_-$ Walls with radius greater than certain size

8 Appendix - SageMath code

```
# Requires extra package:
#! sage -pip install "pseudowalls==0.0.3" --extra-index-url
↪ https://gitlab.com/api/v4/projects/43962374/packages/pypi/simple

from pseudowalls import *

Δ = lambda v: v.Q_tilt()
mu = stability.Mumford().slope
## ####### ############ ##
## SECTION Introduction ##
## ####### ############ ##

v = Chern_Char(3, 2, -2)
u = Chern_Char(1, 0, 0)

alpha = stability.Tilt().alpha
beta = stability.Tilt().beta

coords_range = (beta, -5, 5), (alpha, 0, 5)

charact_curve_plot = (
  implicit_plot(stability.Tilt().degree(u), *coords_range , rgbcolor =
  ↪ "red")
  + implicit_plot(stability.Tilt().degree(v), *coords_range )
  + line([(mu(v),0),(mu(v),5)], linestyle = "dashed", legend_label =
  r"$(3,2\ell,-4\ell^2/2)$")
  + line([(mu(u),0),(mu(u),5)], rgbcolor = "red", linestyle =
  "dashed", legend_label = r"$(1,0,0)$")
  + implicit_plot(stability.Tilt().wall_eqn(u,v)/alpha,
  *coords_range , rgbcolor = "black")
)
charact_curve_plot.xmax(1)
charact_curve_plot.xmin(-2)
charact_curve_plot.ymax(1.5)
charact_curve_plot.axes_labels([r"$\beta$", r"$\alpha$"])
var("m") # Initialize symbol for variety parameter
## ####### ########### ###### ##
## SECTION B.Schmidt's Method ##
## ####### ########### ###### ##

## ####### ########### ##
## SECTION Limitations ##
## ####### ########### ##
```

```
## ####### ########## ##
## SECTION Refinement ##
## ####### ########## ##

# Requires extra package:
#! sage -pip install "pseudowalls==0.0.3" --extra-index-url
↪  https://gitlab.com/api/v4/projects/43962374/packages/pypi/simple

from pseudowalls import *

v = Chern_Char(*var("R C D", domain="real"))
u = Chern_Char(*var("r c d", domain="real"))

Δ = lambda v: v.Q_tilt()
ts = stability.Tilt
var("beta", domain="real")

c_lower_bound = -(
  ts(beta=beta).rank(u)
  /ts().alpha
).expand() + c

var("q", domain="real")
c_in_terms_of_q = c_lower_bound + q
# RENDERED TO LATEX: c_in_terms_of_q
# First Bogomolov-Gieseker form expression that must be non-negative:
bgmlv1 = Δ(v) - Δ(u) - Δ(v-u)
# RENDERED TO LATEX: 0 <= bgmlv1.expand()
bgmlv1_with_q = (
  bgmlv1
  .expand()
  .subs(c == c_in_terms_of_q)
)
# RENDERED TO LATEX: 0 <= bgmlv1_with_q
var("r_alt",domain="real") # r_alt = r - R/2 temporary substitution

bgmlv1_with_q_reparam = (bgmlv1_with_q.subs(r == r_alt +
↪  R/2)/r_alt).expand()

bgmlv1_d_ineq = (
  ((0 >= -bgmlv1_with_q_reparam)/4 + d) # Rearrange for d
  .subs(r_alt == r - R/2) # Resubstitute r back in
  .expand()
)
```

```
bgmlv1_d_lowerbound = bgmlv1_d_ineq.rhs() # Keep hold of lower bound for d
# RENDERED TO LATEX: bgmlv1_d_ineq
# Separate out the terms of the lower bound for d
bgmlv1_d_lowerbound_without_hyp = bgmlv1_d_lowerbound.subs(1/(R-2*r) == 0)

bgmlv1_d_lowerbound_exp_term = (
  bgmlv1_d_lowerbound
  - bgmlv1_d_lowerbound_without_hyp
).expand()

bgmlv1_d_lowerbound_const_term =
↪  bgmlv1_d_lowerbound_without_hyp.subs(r==0)

bgmlv1_d_lowerbound_linear_term = (
  bgmlv1_d_lowerbound_without_hyp
  - bgmlv1_d_lowerbound_const_term
).expand()

# Verify the simplified forms of the terms that will be mentioned in text
var("chbv",domain="real") # symbol to represent ch_1^\beta(v)

assert bgmlv1_d_lowerbound_const_term == (
  (
    # Keep hold of this alternative expression:
    bgmlv1_d_lowerbound_const_term_alt :=
    (
      chbv/2
      + beta*q
    )
  )
  .subs(chbv == v.twist(beta).ch[2])
  .expand()
)

assert bgmlv1_d_lowerbound_exp_term == (
  (
    # Keep hold of this alternative expression:
    bgmlv1_d_lowerbound_exp_term_alt :=
    (
      - R*chbv/2
      - R*beta*q
      + C*q
      - q^2
    )/(R-2*r)
```

```
  )
  .subs(chbv == v.twist(beta).ch[2])
  .expand()
)
# RENDERED TO LATEX: bgmlv1_d_lowerbound_linear_term
# First Bogomolov-Gieseker form expression that must be non-negative:
bgmlv2 = Δ(u)
# RENDERED TO LATEX: 0 <= bgmlv2.expand()
bgmlv2_with_q = (
  bgmlv2
  .expand()
  .subs(c == c_in_terms_of_q)
)
# RENDERED TO LATEX: 0 <= bgmlv2_with_q
bgmlv2_d_ineq = (
  (0 <= bgmlv2_with_q)/2/r # rescale assuming r > 0
  + d # Rearrange for d
).expand()

# Keep hold of lower bound for d
bgmlv2_d_upperbound = bgmlv2_d_ineq.rhs()
# RENDERED TO LATEX: bgmlv2_d_ineq
# Seperate out the terms of the lower bound for d

bgmlv2_d_upperbound_without_hyp = (
  bgmlv2_d_upperbound
  .subs(1/r == 0)
)

bgmlv2_d_upperbound_const_term = (
  bgmlv2_d_upperbound_without_hyp
  .subs(r==0)
)

bgmlv2_d_upperbound_linear_term = (
  bgmlv2_d_upperbound_without_hyp
  - bgmlv2_d_upperbound_const_term
).expand()

bgmlv2_d_upperbound_exp_term = (
  bgmlv2_d_upperbound
  - bgmlv2_d_upperbound_without_hyp
).expand()
# RENDERED TO LATEX: bgmlv2_d_upperbound_const_term
# RENDERED TO LATEX: bgmlv2_d_upperbound_linear_term
```

```
# RENDERED TO LATEX: bgmlv2_d_upperbound_exp_term
# Third Bogomolov-Gieseker form expression that must be non-negative:
bgmlv3 = Δ(v-u)
# RENDERED TO LATEX: 0 <= bgmlv3.expand()
bgmlv3_with_q = (
  bgmlv3
  .expand()
  .subs(c == c_in_terms_of_q)
)
# RENDERED TO LATEX: 0 <= bgmlv3_with_q
var("r_alt",domain="real") # r_alt = r - R temporary substitution

bgmlv3_with_q_reparam = (
  bgmlv3_with_q
  .subs(r == r_alt + R)
  /r_alt # This operation assumes r_alt > 0
).expand()

bgmlv3_d_ineq = (
  ((0 <= bgmlv3_with_q_reparam)/2 + d) # Rearrange for d
  .subs(r_alt == r - R) # Resubstitute r back in
  .expand()
)

# Check that this equation represents a bound for d
assert bgmlv3_d_ineq.lhs() == d

bgmlv3_d_upperbound = bgmlv3_d_ineq.rhs() # Keep hold of lower bound for d
# RENDERED TO LATEX: bgmlv3_d_ineq
# Seperate out the terms of the lower bound for d

bgmlv3_d_upperbound_without_hyp = (
  bgmlv3_d_upperbound
  .subs(1/(R-r) == 0)
)

bgmlv3_d_upperbound_const_term = (
  bgmlv3_d_upperbound_without_hyp
  .subs(r==0)
)

bgmlv3_d_upperbound_linear_term = (
  bgmlv3_d_upperbound_without_hyp
  - bgmlv3_d_upperbound_const_term
).expand()
```

```
bgmlv3_d_upperbound_exp_term = (
  bgmlv3_d_upperbound
  - bgmlv3_d_upperbound_without_hyp
).expand()

# Verify the simplified forms of the terms that will be mentioned in text

var("chbv",domain="real") # symbol to represent ch_1^\beta(v)

assert bgmlv3_d_upperbound_const_term == (
  (
    # keep hold of this alternative expression:
    bgmlv3_d_upperbound_const_term_alt := (
      chbv
      + beta*q
    )
  )
  .subs(chbv == v.twist(beta).ch[2]) # subs real val of ch_1^\beta(v)
  .expand()
)

assert bgmlv3_d_upperbound_exp_term == (
  (
    # Keep hold of this alternative expression:
    bgmlv3_d_upperbound_exp_term_alt :=
    (
      R*chbv
      + (C - q)^2/2
      + R*beta*q
      - D*R
    )/(r-R)
  )
  .subs(chbv == v.twist(beta).ch[2]) # subs real val of ch_1^\beta(v)
  .expand()
)
# RENDERED TO LATEX: bgmlv3_d_upperbound_linear_term
# SUB SECTION Bounds on \texorpdfstring{$r$}{r} #
# ### ####### ###### ## ###################### #

# RENDERED TO LATEX: bgmlv1_d_lowerbound_linear_term
# RENDERED TO LATEX: bgmlv1_d_lowerbound_const_term_alt.subs(chbv == 0)
# RENDERED TO LATEX: bgmlv1_d_lowerbound_exp_term_alt.subs(chbv == 0)
# RENDERED TO LATEX: bgmlv2_d_upperbound_linear_term
# RENDERED TO LATEX: bgmlv2_d_upperbound_const_term
```

```
# RENDERED TO LATEX: bgmlv2_d_upperbound_exp_term
# RENDERED TO LATEX: bgmlv3_d_upperbound_linear_term
# RENDERED TO LATEX: bgmlv3_d_upperbound_const_term_alt.subs(chbv == 0)
# RENDERED TO LATEX: bgmlv3_d_upperbound_exp_term_alt.subs(chbv == 0)
positive_radius_condition = (
  (
    (0 > - u.twist(beta).ch[2])
    + d # rearrange for d
  )
  .subs(solve(q == u.twist(beta).ch[1], c)[0]) # express c in term of q
  .expand()
)
# RENDERED TO LATEX: positive_radius_condition
def beta_min(chern):
  ts = stability.Tilt()
  return min(
    map(
      lambda soln: soln.rhs(),
      solve(
        (ts.degree(chern))
          .expand()
          .subs(ts.alpha == 0),
        beta
      )
    )
  )

v_example = Chern_Char(3,2,-2)
q_example = 7/3

def plot_d_bound(
  v_example,
  q_example,
  ymax=5,
  ymin=-2,
  xmax=20,
  aspect_ratio=None
):

  # Equations to plot imminently representing the bounds on d:
  eq1 = (
    bgmlv1_d_lowerbound
    .subs(R == v_example.ch[0])
    .subs(C == v_example.ch[1])
    .subs(D == v_example.ch[2])
```

```
  .subs(beta = beta_min(v_example))
  .subs(q == q_example)
)

eq2 = (
  bgmlv2_d_upperbound
  .subs(R == v_example.ch[0])
  .subs(C == v_example.ch[1])
  .subs(D == v_example.ch[2])
  .subs(beta = beta_min(v_example))
  .subs(q == q_example)
)

eq3 = (
  bgmlv3_d_upperbound
  .subs(R == v_example.ch[0])
  .subs(C == v_example.ch[1])
  .subs(D == v_example.ch[2])
  .subs(beta = beta_min(v_example))
  .subs(q == q_example)
)

eq4 = (
  positive_radius_condition.rhs()
  .subs(q == q_example)
  .subs(beta = beta_min(v_example))
)

example_bounds_on_d_plot = (
  plot(
    eq3,
    (r,v_example.ch[0],xmax),
    color='green',
    linestyle = "dashed",
    legend_label=r"upper bound: $\Delta(G) \geq 0$",
  )
  + plot(
    eq2,
    (r,0,xmax),
    color='blue',
    linestyle = "dashed",
    legend_label=r"upper bound: $\Delta(E) \geq 0$"
  )
  + plot(
    eq4,
```

```
      (r,0,xmax),
      color='orange',
      linestyle = "dotted",
      legend_label=r"lower bound: $\mathrm{ch}_2^{\beta_{-}}(E)>0$"
    )
    + plot(
      eq1,
      (r,v_example.ch[0]/2,xmax),
      color='red',
      linestyle = "dotted",
      legend_label=r"lower bound: $\Delta(E) + \Delta(G) \leq \Delta(F)$"
    )
  )
  example_bounds_on_d_plot.ymin(ymin)
  example_bounds_on_d_plot.ymax(ymax)
  example_bounds_on_d_plot.axes_labels(['$r$', '$d$'])
  if aspect_ratio:
    example_bounds_on_d_plot.set_aspect_ratio(aspect_ratio)
  return example_bounds_on_d_plot

var("a_F b_q n") # Define symbols introduce for values of beta and q
beta_value_expr = (beta == a_F/n)
q_value_expr = (q == b_q/n)
# RENDERED TO LATEX: positive_radius_condition
↪   .subs([q_value_expr,beta_value_expr]).factor()
var("epsilon")

# Tightness conditions:

bounds_too_tight_condition1 = (
  bgmlv2_d_upperbound_exp_term
  < epsilon
)

bounds_too_tight_condition2 = (
  bgmlv3_d_upperbound_exp_term_alt.subs(chbv==0)
  < epsilon
)
# RENDERED TO LATEX: bgmlv2_d_upperbound_exp_term
# RENDERED TO LATEX: bgmlv3_d_upperbound_exp_term_alt.subs(chbv==0)
# rearrange the "tightness" conditions in terms of r

bounds_too_tight_condition1 = (
  (bounds_too_tight_condition1 * r / epsilon)
  .expand()
```

```
)
bounds_too_tight_condition2 = (
  (bounds_too_tight_condition2 * (r - R) / epsilon + R)
  .expand()
)

# Check that these are indeed rearranged for r
assert bounds_too_tight_condition1.rhs() == r
assert bounds_too_tight_condition2.rhs() == r
# RENDERED TO LATEX: c_in_terms_of_q.subs([q_value_expr,beta_value_expr])
rhs_numerator = (
  positive_radius_condition
  .rhs()
  .subs([q_value_expr,beta_value_expr])
  .factor()
  .numerator()
)
# RENDERED TO LATEX: positive_radius_condition
↪   .subs([q_value_expr,beta_value_expr]).factor()
## ####### ########## ##
## SECTION Conclusion ##
## ####### ########## ##

## ####### ######## # ######## #### ##
## SECTION Appendix - SageMath code ##
## ####### ######## # ######## #### ##
```